Directed and Undirected Graphs

In this activity you will design both directed and undirected graphs to solve problems from topics such as transportation and chess.

So what exactly are "directed graphs" and "undirected graphs"?

In the context of equations in two variables like:

$$x^2 + y^2 = 16$$

The *graph* of this equation is a visual depiction of its solutions. (See Figure 1.)

Note that each point on the depicted circle has coordinates of the form (x,y) which satisfy the given equation. For example the points (4,0), (-4,0), (0,4), and (0,-4) all satisfy $x^2 + y^2 = 16$ and thus lie on the equation's graph.

In fact, this circle is composed of an infinite number of points of the form (x,y), each of which satisfies the equation $x^2 + y^2 = 16$.

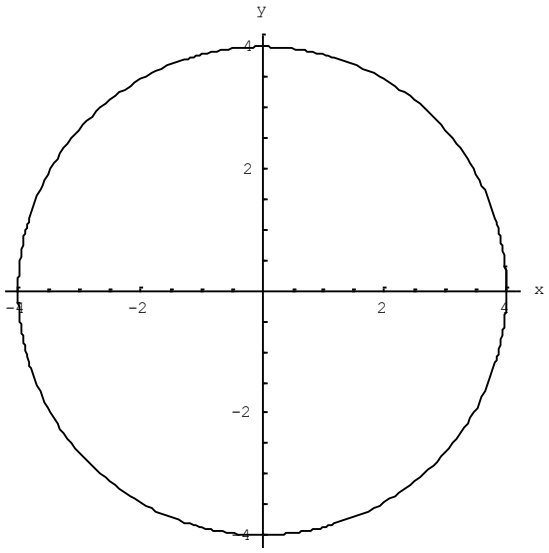
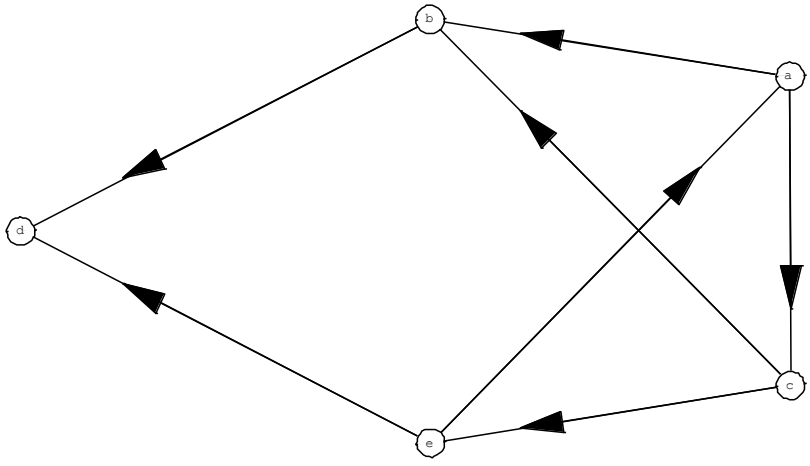


Figure 1.

Quite often it is useful to consider graphs outside the context of equations. Here the "points" on the graphs are often referred to as *vertices*. Such a *directed graph* or *digraph* consists of a finite, nonempty set of vertices **V** and a finite set **E** of ordered pairs of distinct elements of **V**.

The directed graph depicted in Figure 2. has a set of vertices $\mathbf{V} = \{a, b, c, d, e\}$ and a set of ordered pairs $\mathbf{E} = \{(a,b), (a,c), (b,d), (c,b), (c,e), (e,a), (e,d)\}$



Here the directed arrows indicate a "one-way" flow in the graph. For example there is a direct path from vertices a to b however there is no such path from vertex b to vertex a.

Figure 2.

Below is the Mathematica code that produced the directed graph depicted in Figure 2.

```
<<DiscreteMath`GraphPlot`;

h={a→b,a→c,b→d,c→b,c→e,e→a,e→d};

coord=GraphCoordinates[h];
labels=VertexList[h];

esf[i_,j_]:=Block[{},{Black,Line[{i,j}],Arrow[coord[[i]],
coord[[i]]+0.75(coord[[j]]-coord[[i]])]}];

GraphPlot[h,VertexStyleFunction→({White,Disk[#,0.03],Black,Circle[#,0.0
3],Black,Text[labels[[#]],#]}&),VertexCoordinates→coord,EdgeStyleFuncti
on→esf];
```

Problem 1.

(a) Use the provided code to make a directed graph with the given information:
Vertices: $V = \{1, 2, 3, 4, 5\}$
Ordered Pairs: Each vertex is paired with all other vertices.

(Such a graph is said to be a "complete graph")

(b) Make an adjacency matrix for this graph.

(c) How many different paths of length 3 are there from vertex 1 to vertex 5?

Undirected graphs have no specified flow between the vertices. This may be fitting for transport problems where all edges are "two-way" edges. Did you know that in 1736, Swiss mathematician Leonhard Euler solved such a problem in graph theory that had been puzzling the good citizens of the town of Konigsberg in Prussia (now Kaliningrad in Russia). The river Pregel divides the town into four sections, and in Euler's days seven bridges connected these sections. The people wanted to know if it were possible to start

at any location in town, cross every bridge exactly once, and return to the starting location. Euler showed that it is impossible to take such a walk. (Kolman 290)

Problem 2.

A small town has 6 buildings: a school, a sheriff's office, a general store, a gas station, a diner, and a post office. You can get to all buildings directly from the general store. There are roads from the school to both the post office and the diner. There are roads from the gas station to both the diner and the sheriff's office. There are three roads leading to the post office. The town has 10 roads. Make an undirected graph of the town.

Is there more than one possible layout for this town? (Explain)

Does this graph have an Euler path?

Did you know that today, chess is one of the world's most popular games, played by an estimated 605 million people worldwide in clubs, online, by correspondence (mail and e-mail), in tournaments (amateur and professional) and informally. Aspects of art and science are found in chess composition and theory. Chess is also advocated as a way of enhancing mental prowess. The game is played on a square chequered chessboard. At the start, each player ("white" and "black") controls sixteen pieces: one king, one queen, two rooks, two knights, two bishops, and eight pawns. The object of the game is to checkmate the opponent's king, whereby the king is under immediate attack (in "check") and there is no way to remove it from attack on the next move. Theoreticians have developed extensive chess strategies and tactics since the game's inception.

The tradition of organized competitive chess started in the 16th century. The first official World Chess Champion, Wilhelm Steinitz, claimed his title in 1886; Vladimir Kramnik, today, is the 14th Champion in this lineage. There are also biennial world team events called Chess Olympiads. Since the 20th century, two international organizations, the World Chess Federation and the International Correspondence Chess Federation have organized and overseen the top chess competitions and international titles. One of the goals of early computer scientists was to create a chess-playing machine, and today's

chess is deeply influenced by the abilities of current chess programs. In 1997, a match between Garry Kasparov, then World Champion, and IBM's Deep Blue chess program proved for the first time that computers are able to beat even the strongest human players. The popularity of online chess coincided with the growth of the Internet, which started in the mid 1990s. (WIKIPEDIA)

Let's use directed graphs to help find the best line of play for white in the position below.

White to play and win!

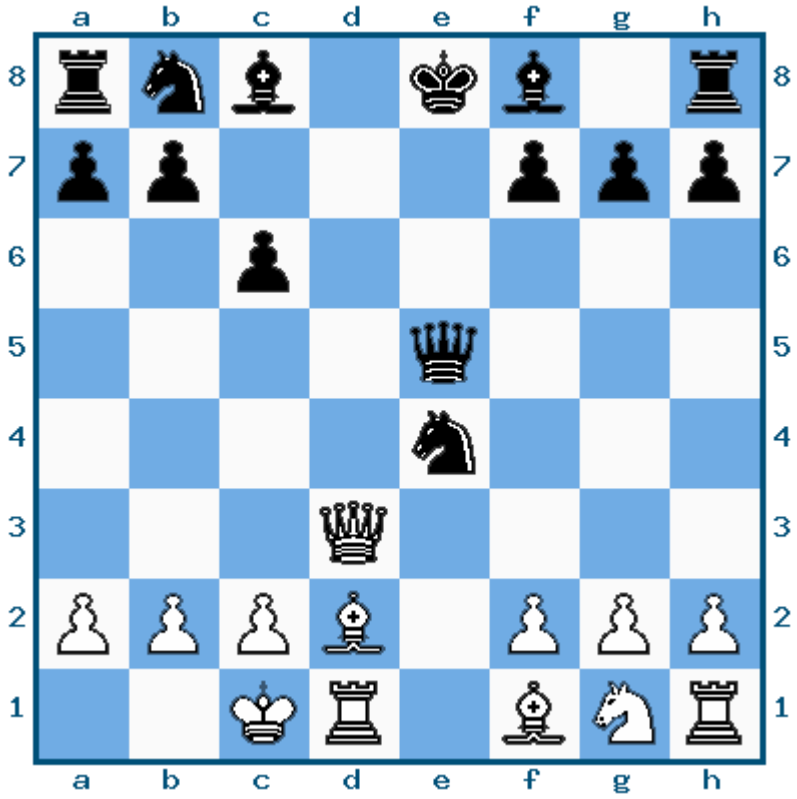


Figure 3.

Problem 3.
Make a directed graph with a vertex labeled "cp" for current position. The graph should contain the ordered pairs: (cp, 1Ba5), (cp,1Bf4), (cp,1Bg5), (cp,1Nf3), (cp,1Qd8+), as well as the ordered pairs representing black's responses to white's candidate moves, white's responses to black's responses, and so on. When you deem a variation to lose for white, indicate this on your directed graph with a vertex labeled "bw" for black wins. Your graph should also have a vertex "ww" for white wins. Can you find a path from "cp" to "ww"?

Did you make any assumptions in finding the solution to this puzzle?

Can you think of any deficiencies in the method used to solve this puzzle?