Matrices

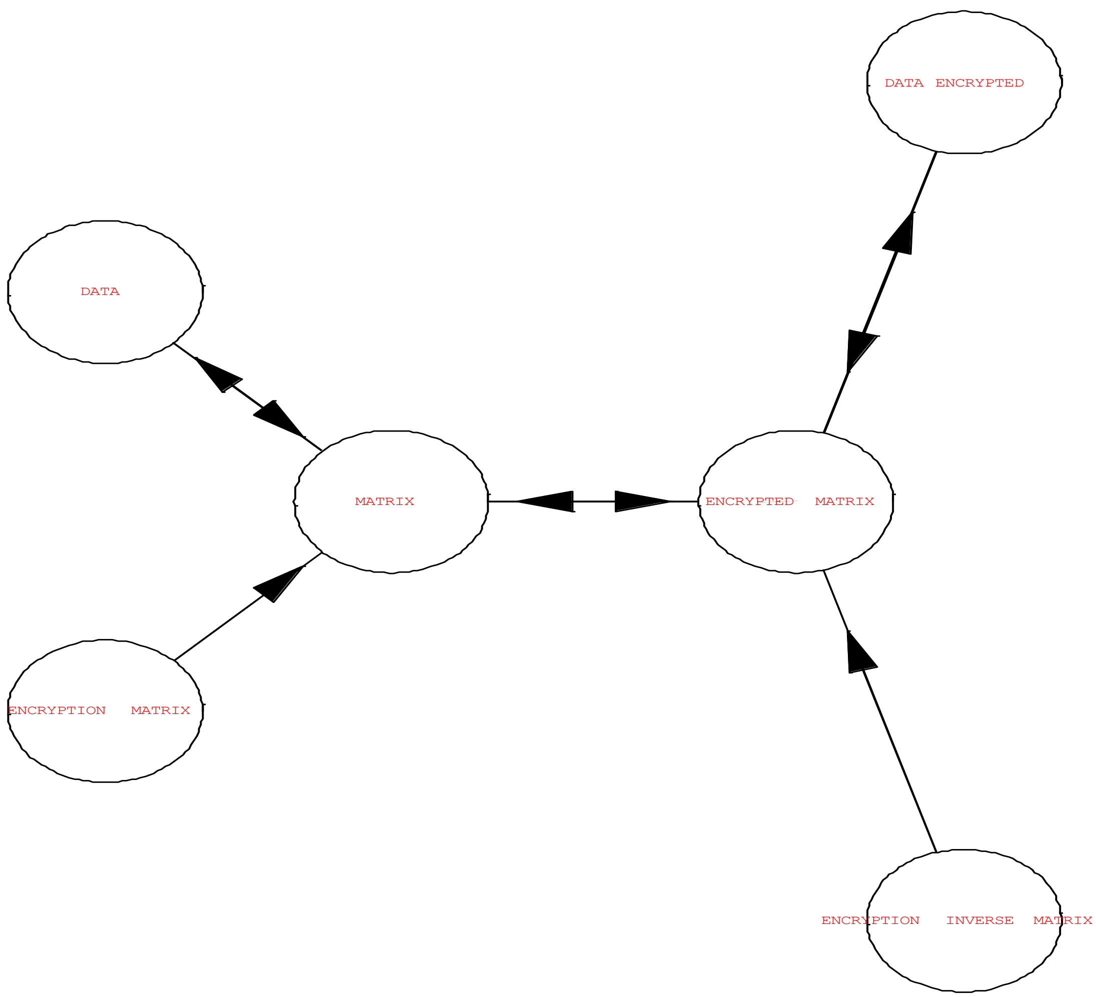
Recall that a matrix is a rectangular array of numbers, the size of which is determined by its number of rows and columns. $\underset{2x1}{A} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\underset{2x2}{B} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\underset{3x4}{C} = \begin{bmatrix} 1 & -2 & 7 & .5 \\ 3 & 0 & 1 & .2 \\ 7 & .4 & 15 & 0 \end{bmatrix}$

The notation $\underset{mxn}{A}$ specifies that matrix A has m rows and n columns. Conventionally uppercase letters specify matrices while subscripted lowercase letters refer to the individual entries of these matrices. For example, the entry c_{13} refers to entry of matrix C in the 1st row & 3rd column. Thus $c_{13} = 7$ while $b_{21} = 0$ and $a_{11} = 1$.

What can we say of a matrix $\underset{mxm}{A}$ for which $a_{ij} = a_{ji}$ for all $i, j \in N$ where $i, j \leq m$ **???**

Matrices have diverse applications in pure and applied mathematics. Matrix methods abound to solve systems of linear equations, difference equations, and differential equations. Additionally, matrices can serve as tools in diverse fields such as image processing, electrical circuit analysis, and cryptography.

The following flow chart shows how matrices can be used to "encrypt" information.



For example, an ATM pin typically has 4 digits. To encrypt the pin: 5279

STEP 1: Convert the data to a matrix form. e.g. $5279 \rightarrow \begin{bmatrix} 5 & 2 \\ 7 & 9 \end{bmatrix}$

STEP 2: Pre-multiply by a "cipher" (an invertible encryption matrix) like: $\begin{bmatrix} 1 & 3 \\ 0 & 5 \end{bmatrix}$

$$\text{e.g.} \quad \begin{bmatrix} 1 & 3 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 5 & 2 \\ 7 & 9 \end{bmatrix} = \begin{bmatrix} 26 & 29 \\ 35 & 45 \end{bmatrix}$$

STEP 3: Convert the encrypted matrix to encrypted data. e.g. $\begin{bmatrix} 26 & 29 \\ 35 & 45 \end{bmatrix} \rightarrow 26293545$

Without the cipher, our ATM pin may be lost forever!

STEP 4: To decrypt the encrypted data, post-multiply by the encryption matrix inverse.

$$\text{e.g.} \quad \begin{bmatrix} 26 & 29 \\ 35 & 45 \end{bmatrix} \begin{bmatrix} 1 & -.6 \\ 0 & .2 \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ 7 & 9 \end{bmatrix} \quad \text{and now} \quad \begin{bmatrix} 5 & 2 \\ 7 & 9 \end{bmatrix} \rightarrow 5279$$

We have recovered the ATM pin!

Problem 1.

(a) Encrypt the ATM pin #: 3856 using the cipher: $\begin{bmatrix} -2 & 5 \\ 3 & 6 \end{bmatrix}$

(b) Devise a matrix method to encrypt the message: "***The cat walks at midnight***"

(c) Why must the encryption matrix be invertible? Would the identity matrix make for a good encryption matrix (why or why not)?

(d) Given the encrypted matrix $\begin{bmatrix} 48 & 6 & 79 & 73 \\ 40 & 5 & 57 & 68 \\ 13 & 4 & 30 & 54 \\ 4 & 1 & 25 & 27 \end{bmatrix}$ & the cipher $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$,

Can you decode the hidden message? (Hint: $a = 1, b = 2, c = 3, \ldots$)

Eigenvalue History

Presently, eigenvalues are often introduced in the context of matrix theory. Historically, however, they arose in the study of quadratic forms and differential equations. In the first half of the 18th century, Johann and Daniel Bernoulli, d'Alembert and Euler encountered eigenvalue problems when studying the motion of a rope, which they considered to be a weightless string loaded with a number of masses. Laplace and Lagrange continued their work in the second half of the century. They realized that the eigenvalues are related to the stability of the motion. They also used eigenvalue methods in their study of the solar system.

Euler had also studied the rotational motion of a rigid body and discovered the importance of the principal axes. As Lagrange realized, the principal axes are the eigenvectors of the inertia matrix. In the early 19th century, Cauchy saw how their work could be used to classify the quadric surfaces, and generalized it to arbitrary dimensions. Cauchy also coined the term *racine caractéristique* (characteristic root) for what is now called *eigenvalue*; his term survives in *characteristic equation*.

Fourier used the work of Laplace and Lagrange to solve the heat equation by separation of variables in his famous 1822 book *Théorie analytique de la chaleur*. Sturm developed Fourier's ideas further and he brought them to the attention of Cauchy, who combined them with his own ideas and arrived at the fact that symmetric matrices have real eigenvalues. This was extended by Hermite in 1855 to what are now called Hermitian matrices. Around the same time, Brioschi proved that the eigenvalues of orthogonal matrices lie on the unit circle, and Clebsch found the corresponding result for skew-symmetric matrices. Finally, Weierstrass clarified an important aspect in the stability theory started by Laplace by realizing that defective matrices can cause instability.

At the start of the 20th century, Hilbert studied the eigenvalues of integral operators by considering them to be infinite matrices. He was the first to use the German word *eigen* to denote eigenvalues and eigenvectors in 1904, though he may have been following a related usage by Helmholtz. "Eigen" can be translated as "own", "peculiar to", "characteristic" or "individual"—emphasizing how important eigenvalues are to defining the unique nature of a specific transformation. For some time, the standard term in English was "proper value", but the more distinctive term "eigenvalue" is standard today.

The first numerical algorithm for computing eigenvalues and eigenvectors appeared in 1929, when Von Mises published the power method. One of the most popular methods today, the QR algorithm, was proposed independently by Francis and Kublanovskaya in 1961. (WIKIPEDIA)

Have you ever seen the video of the collapse of the Tacoma Narrows Bridge?

The Tacoma Bridge was built in 1940. From the beginning, the bridge would form small waves like the surface of a body of water. This accidental behavior of the bridge brought many people who wanted to drive over this moving bridge. Most people thought that the bridge was safe despite the movement. However, about four months later, the oscillations (waves) became bigger. At one point, one edge of the road was 28 feet higher than the other edge. Finally, this bridge crashed into the water below. One explanation for the crash is that the oscillations of the bridge were caused by the frequency of the wind being too close to the natural frequency of the bridge. The natural frequency of the bridge is the eigenvalue of smallest magnitude of a system that models the bridge. This is why eigenvalues are very important to engineers when they analyze structures. (*Differential Equations and Their Applications*, 1983, pp. 171-173).

Problem 2.

(a) Is $\lambda = 2$ an eigenvalue of $\begin{bmatrix} 3 & 2 \\ 3 & 8 \end{bmatrix}$? Why or why not?

(b) Is $\begin{bmatrix} 4 \\ -3 \\ 1 \end{bmatrix}$ an eigenvector of $\begin{bmatrix} 3 & 7 & 9 \\ -4 & -5 & 1 \\ 2 & 4 & 4 \end{bmatrix}$? If so, find the eigenvalue.

(c) Use a computational engine to find the eigenvalues and eigenvectors of the matrices:

$$\begin{bmatrix} 3 & 2 \\ 3 & 8 \end{bmatrix}, \quad \begin{bmatrix} 3 & 7 & 9 \\ -4 & -5 & 1 \\ 2 & 4 & 4 \end{bmatrix}, \text{ and } \begin{bmatrix} 4 & -9 & -7 & 8 & 2 \\ -7 & -9 & 0 & 7 & 14 \\ 5 & 10 & 5 & -5 & -10 \\ -2 & 3 & 7 & 0 & 4 \\ -3 & -13 & -7 & 10 & 11 \end{bmatrix}$$